Black Hole Radiation Properties Extracted from Quantum Scattering Amplitudes

A Thesis Presented in Partial Fulfillment of
the Honors in the Discipline
in the Subject of Physics

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Abstract

As gravitational wave astronomy advances, there is a growing need for efficient models to describe black hole mergers. While classical methods like Numerical Relativity are computationally expensive, modern Quantum Field Theory (QFT) offers a powerful alternative. In this work we explore how to extract classical gravitational waves directly from quantum scattering amplitudes. Using the Kosower-Maybee-O'Connell (KMOC) formalism, this work first demonstrates how classical fields emerge from coherent quantum states in electrodynamics before extending the framework to gravity. Then the details of the derivation of the observable gravitational spectrum is generated by evaluating the Riemann curvature tensor between S-matrix evolved states. We then evaluate the graviton field operator, uncovering a key dimensional scaling issue when applying the classical Transverse-Traceless (TT) gauge to the quantum limit. Ultimately, this thesis discusses the gap between classical Black Hole Perturbation Theory and cutting-edge quantum techniques, offering an accessible guide to the successes and open challenges of modeling black hole gravitational radiation.

Contents

Acknowledgments

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Citations to Presented Work

Parts of this thesis cover material presented in the following works:

- Aleyna Koro, Andrew Laeuger, Colin Weller, and Yanbei Chen. Imprints of the Frequency-Domain Source Function on Black Hole Ringdown. *LIGO Document* T2500247-v1. [13]
- Aleyna Koro, Andrew Laeuger, Colin Weller, and Yanbei Chen. Imprints of the Frequency-Domain Source Function on Black Hole Ringdown. *American Physical Society, APS New England Section (NES), Annual Meeting*, Providence, Rhode Island, November 2025. Abstract C02.00012 [14]
- Aleyna Koro. Black hole scattering connects general relativity and quantum mechanics. *RISE Conference, Northeastern University*, Boston, April 2023. [11]
- Aleyna Koro. Black Holes and Gravitational Waves: Imprints of the Frequency-Domain Source on Black Hole Ringdown. *Society of Physics Students, Northeastern University*, Boston, September 2025. [12]

1 Introduction

1.1 Project Overview

Black holes sit at the extreme intersection of General Relativity and Quantum Mechanics, making them ideal subjects for studying the fundamental connections between the two theories. With the first direct detection of a black hole merger in 2015, modeling these events has transitioned from a purely theoretical pursuit to a crucial observational science. However, the mathematics and physics required to fully understand them are almost exclusively reserved for advanced graduate coursework. This leaves a significant gap in the literature for younger students.

This thesis is primarily written for advanced undergraduates to early graduate students in physics, aiming to bridge that gap. By exploring electromagnetic and gravitational radiation, this work connects Black Hole Perturbation Theory to quantum scattering amplitude techniques. This document is structured as follows. Sec. 2 introduces the standard classical framework for black hole radiation. Sec. 3 shifts the focus to quantum mechanics and scattering amplitude techniques. Sec. 4 connects gravity to quantum mechanics by introducing the Double Copy framework and applying the quantum tools developed in Sec. 3 to the gravitational field. Finally, Sec. 5 provides an outlook on the future of this field.

1.2 My Undergraduate Experience during this Project

This work involves various physics concepts that would require years to fully master. To navigate this steep learning curve, my advisor Professor Camille Gómez-Laberge and I employed the pedagogical technique of just-in-time learning. This technique is characterized by picking only the necessary topics needed to understand an equation or concept. Ultimately, I hope this thesis serves as proof that the JITL method is an incredibly effective tool, especially for undergraduate students in the early stages of their research careers. By tackling complex physics as the need arises allows you to build a physical intuition that makes formal advanced coursework much less intimidating when you eventually encounter it. My goal is for this work to immerse you in the evolving field of gravitational physics and to show that while the mathematics can be daunting, it is accessible. There is still a tremendous amount of work to be done in bridging the classical and quantum regimes and I hope this text gives you the confidence to be a part of it.

2 Black Hole Perturbation Theory

2.1 Perturbation Theory Basics

2.1.1 The Metric

An important concept for black hole physics classically and in quantum mechanics is the metric. The metric encodes the geometry of spacetime. The metric defines distances and time intervals in our universe. Locally spacetime is flat. It is described using the Minkowski metric, $\eta_{\alpha\beta}$. The Minkowski metric is written in this matrix form shown in Eqn. 2.1. In this work, we adopt the $(-+++)$ metric signature convention, defining the flat Minkowski metric, unless stated otherwise. It is important to note that the opposite convention $(+---)$ is also widely used in the literature.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

Globally, the presence of mass and energy induces curvature across the spacetime geometry. In Black Hole Perturbation Theory (BHPT), this global curvature is captured by defining a background metric, $\bar{g}_{\alpha\beta}$. The background metric $\bar{g}_{\alpha\beta}$ describes exact solutions to the Einstein Field Equations, such as a stationary (Schwarzschild) or a rotating (Kerr) black hole. To account for gravitational radiation, a small linear perturbation ($h_{\alpha\beta}$) to the background metric is introduced. The full spacetime metric is therefore written in the form $g_{\alpha\beta}(x) = \bar{g}_{\alpha\beta}(x) + h_{\alpha\beta}(x)$. While $\bar{g}_{\alpha\beta}$ governs the strongly warped space in the immediate vicinity of a black hole, it asymptotically approaches the flat Minkowski limit, $\eta_{\alpha\beta}$, far away from any massive objects. For a distant observer measuring a binary merger, the background is effectively flat, allowing the perturbation $h_{\alpha\beta}$ to be treated as a freely propagating wave.

2.1.2 Teukolsky Equation and Waveform

BHPT treats gravity as a linear field on a fixed background, allowing us to approximate the complex and nonlinear Einstein Field Equations. This framework provides the benchmarks for Numerical Relativity(NR) and creates a mathematical bridge to the particle-like descriptions of gravity used in scattering amplitudes as we will see later on.

The Teukolsky equation governs the dynamics of linear perturbations on a Kerr background acting as the generalized wave equation for a rotating black hole. The radial part tells us how the wave travels out to the observer(e.g. LIGO), and the angular part tells us which way the black hole is pointing.

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) \quad +\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right)R=0 \tag{2.2}$$

The function R(r) is the radial part of the perturbation. s is the spin weight of the perturbation, which is $s=2$ for gravity. $\Delta=r^2-2Mr+a^2$ is the horizon function of the Kerr metric, where M is the black hole mass and a is its spin parameter. The function $K(r)=(r^2+a^2)\omega-am$, where ω is the Fourier frequency and m is the azimuthal quantum number.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}-2a\omega s\cos\theta-\frac{2ms\cos\theta}{\sin^2\theta}-s^2\cot^2\theta+s+A\right)S=0 \tag{2.3}$$

The frequency-domain solution $R(r)$ is converted to the time domain via an inverse Fourier transform, yielding $\psi(t,r)_{\text{hom}}$ from Eqn. 2.4, the time-domain Teukolsky radial function. In our case, $\psi(t,r)_{\text{hom}}$ is the direct output of the Green's-function inversion. The Green's function in this physical context is the response of the black hole spacetime to a source, $S_{\ell m}(r'_*,\omega)$.

$$\psi(t,r_*)_{\text{hom}}=\sum_{\ell,m}\int\frac{d\omega}{2\pi}\,e^{-i\omega t}\int dr'_*G_{\ell m}(r_*,r'_*,\omega)\,S_{\ell m}(r'_*,\omega). \tag{2.4}$$

For more detailed information on BHPT refer to [16, 6].

2.2 Black Hole Mergers

Gravitational waves that we detect from Earth are largely produced by merging black holes. To understand the emitted gravitational waves from these mergers, we first need to understand the different stages of the mergers.

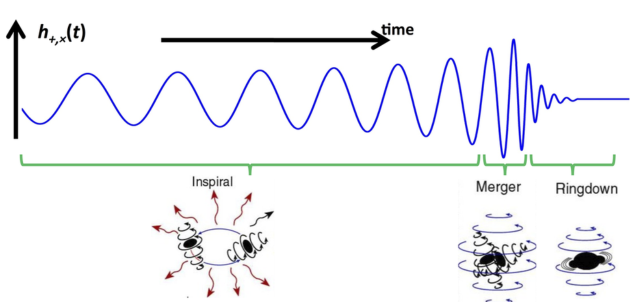

Figure 1: Black Hole merger phases

Fig. 1 shows the gravitational wave strain over time split into three phases. The first phase is the inspiral where the black holes (or black hole and neutron star) are orbiting each other, slowly spiraling into each other. This phase emits low frequency at a longer period of time compared to the other phases. The next phase is the merger, which as the name suggest is the moment the black holes combine into each other. The frequency is higher, but the duration of this phase is very short. Finally, the ringdown is the phase where the remnant black hole after merger oscillates as it settles into its new state as a spinning black hole.

2.2.1 Quasinormal Modes and the No-Hair Theorem

The ringdown is characterized by discrete, complex-valued frequencies known as Quasinormal Modes(QNMs). QNMs are damped by the loss of energy to gravitational radiation. A QNM is described by $\omega = \omega_R + i\omega_I$, where the real part represents the oscillation frequency and the imaginary part represents the decay rate. The importance of QNMs lies in the No-Hair Theorem [10]. According to General Relativity, a stationary black hole is described entirely by just two parameters: its mass (M) and its spin (a). Therefore, the entire spectrum of QNMs is determined solely by these two values. QNMs are used as a test of General Relativity to make sure that the black hole mergers measured by LIGO are in agreement.

2.2.2 Modeling the Ringdown as a Scattering Problem

To model the ringdown analytically, we treat it as a perturbation to the remnant black hole. This perturbation could be caused by the merger or an infalling mass occurring near the black hole. The ringdown we detect is the gravitational wave scattering off the photon ring around the final black hole and propagating to an observer at infinity.

We illustrate this using a toy model in Fig. 2 [13]. We define a source S as a Gaussian wave packet that appears in the radial domain. We then solve the radial Teukolsky equation (Eqn. 2.2) to see how this signal evolves.

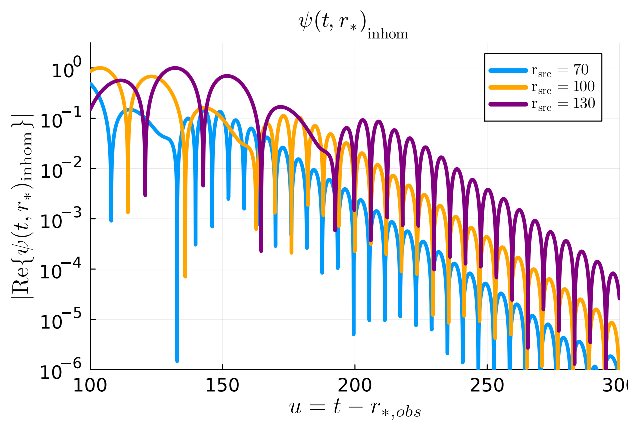

Figure 2: Example of Black Hole Ringdown Toy Model

Since the black hole is unperturbed prior to the disturbance of the source, the waveform must vanish for t < 0. Physically, no radiation can be observed before the perturbation occurs.

2.3 Limitations

While BHPT and NR have allowed us to detect and understand the first encounters of gravitational wave data, both methods face significant hurdles as we build more precise detectors that require more accurate models. NR involves solving the full, non-linear Einstein Field Equations on supercomputers. While this is the only way to model the exact moment of a merger, it is incredibly expensive in terms of time and resources. A single simulation of two black holes merging can take weeks or even months of processing. There are also too many combinations of black hole masses and spins. Since NR is so slow, we cannot simulate every possible scenario, leaving gaps in the templates we use to identify signals from LIGO data. BHPT is much faster than NR because it uses a linear approximation ($g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$). However, BHPT only works when the deviation($h_{\mu\nu}$) to the background ($\bar{g}_{\mu\nu}$) is much smaller than the background because that is what allows us to only keep the linear terms. BHPT cannot capture the nonlinearities that occur with strong gravity. Therefore, we need a more developed yet less expensive and quicker way to have more accurate models.

This is where using QFT techniques may be useful. As recent research in high-energy physics has shown, scattering amplitudes have revealed mathematical structures in gauge theory and gravity, leading to new physical insights, efficient methods for computation, and potential advancements in developing a quantum gravity theory [1, 4]. It is this potential for a more efficient and fundamental understanding of gravity that motivates the transition into the quantum framework explored in the following sections.

3 Quantum Mechanics

3.1 Quantum Scattering Amplitudes

When particles interact or scatter, the scattering amplitude is the quantity that describes the process from the initial state to the final state of the interaction. For example, two electrons scatter off each other and emit a photon. These processes can be encoded into a scattering amplitude that includes the momentum, polarization, and helicity(projection of particle's spin onto its direction of motion) of each particle involved. For example, the electromagnetic field operator is described as:

$$\mathbb{A}_\mu(x) = \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(k) \left[a_{(\eta)}(k) \varepsilon_\mu^{(\eta)*}(k) e^{-ikx/\hbar} + a_{(\eta)}^\dagger(k) \varepsilon_\mu^{(\eta)}(k) e^{+ikx/\hbar} \right] \tag{3.1}$$

where η is the helicity, $a_{(\eta)}(k)$ and $a_{(\eta)}^\dagger(k)$ are the annihilation and creation operators respectively, ε are the polarizations, and k is the momentum [19]. In this expression and in what follows, we set the speed of light to $c = 1$ but keep Planck's constant as $\hbar = 1.054 \times 10^{-34} \mathrm{J} \cdot \mathrm{s}$ in order to track quantum effects explicitly.

For a gauge theory (like Electromagnetism), the scattering amplitude $\mathcal{A}$ for m particles at L loops is expressed as [17]:

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{\ell=1}^{L} \frac{d^d p_\ell}{(2\pi)^d} \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

(3.2)

Where g is the coupling constant that quantifies the interaction strength between particles. The denominator represents the propagators($p_{\alpha i}^2$). This encodes how a virtual messenger particle (like a photon or graviton) travels from the initial state to the final state. The kinematic numerator(n_i) contains the information from our field operators. It contains the momenta (p) and the polarizations (ε) of the particles. c_i is the color factor which is just the charge in electromagnetism. We sum over all possible momenta in this interaction.

3.1.1 Annihilation and Creation Operators

A quantum field is a physical property that exists at every single point in space simultaneously. Fields are also our way of mathematically modeling space-time. We have the vacuum state, denoted by $|0\rangle$ that represents a region of space with no particles. To describe the presence of a particle, we look to the mathematics of the quantum harmonic oscillator. We define two fundamental operators that act upon the field whether it be empty or not:

- **The Creation Operator ($\hat{a}_\eta^\dagger(k)$):** Acting on the vacuum, this operator creates a single particle with momentum k and helicity η.

$$\hat{a}_\eta^\dagger(k) \, |0\rangle = |k, \eta\rangle$$

- **The Annihilation Operator ($\hat{a}_\eta(k)$):** This operator removes a particle, returning that specific mode of the field to the ground state.

$$\hat{a}_\eta(k) \, |k, \eta\rangle = |0\rangle$$

These ladder operators as we call them allow us to populate our spacetime with any number of particles. When we consider a gravitational wave, we are essentially describing a state where a nearly infinite number of these particles overlap to form a smooth, classical wave.

3.1.2 S-Matrix and Feynman Diagrams

The S-matrix (Scattering matrix) represents the total evolution of a quantum system from the infinite past to the infinite future. In any scattering event, we can mathematically decompose this process into two parts:

$$S = 1 + iT$$

The Identity (1): This term represents the case where no interaction occurs. The initial state of the particles or fields remains completely unchanged. In this scenario, the particles pass through the system as if no other particles or fields were present. The Transition Matrix (iT): This term represents the actual interaction. It captures the transition from the initial state to a different final state. Any change in momentum, the emission of radiation, or the merging of two objects is contained entirely within this T operator.

When we calculate a scattering amplitude, we are specifically calculating an element of the T-matrix. Each Feynman diagram like Fig. 3 is a visual representation of a mathematical term within the expansion of T. By isolating T from S, we ignore the empty background and focus solely on the dynamics of the interaction that produces the waveform or particle.

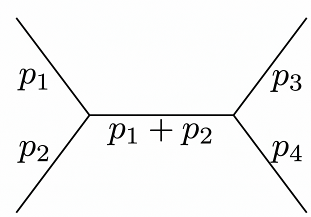

Figure 3: Feynman Diagram

3.1.3 Propagators

To calculate how an interaction at one point in space affects another, we use the propagator. Physically, the propagator describes how a messenger particle (like photons) travels between two points. The propagator is the Green's function for the field's equation of motion paralleling the Green's function approach used to solve the Teukolsky equation in Eqn. 2.4. The most common form is the Feynman Propagator (D_F) in Eqn. 3.3, which represents the propagation of virtual particles that carry energy, momentum, and charge between interaction at point x to point y in space.

$$D_F(x, y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{ik(x-y)} \tag{3.3}$$

In the scattering processes discussed in Sec. 4, the propagator is responsible for carrying the momentum transfer(q) between the external particles. The messenger in the internal line of Fig. 3 is off-shell, meaning it does not satisfy the energy-momentum relation $E^2 = \vec{k}^2 + m^2$, which requires $k^2 = m^2$. This off-shell exchange is what mediates the gravitational or electromagnetic force. Furthermore, the $+i\epsilon$ prescription in the denominator ensures causality, a property we will rely on in Section 4.2 when we isolate the outgoing retarded radiation from the advanced signals.

3.2 Coherent States

Coherent states are special states that resemble the behavior of a classical harmonic oscillator [7]. They were originally developed for optical physics to describe the quantum state of laser light. Since then, they have had wide applications to other fields of physics such as gravitational physics. Coherent states are a natural bridge between classical and quantum physics.

A primary reason coherent states serve as the limit of classical physics is their relationship to the Heisenberg Uncertainty Principle. While all quantum states have uncertainty, coherent states are minimum uncertainty states that distribute this uncertainty equally [7, 8] between position and momentum.

$$\Delta x \Delta p = \frac{\hbar}{2}$$

Since the uncertainty is minimized and remains constant over time, it follows classical equations as close as the laws of quantum mechanics allow. In this thesis, coherent states allows us to treat infinite number of photons or gravitons as a single coherent and classical wave.

To construct a coherent state, we define the displacement operator $C_{\alpha,(\eta)}$, which acts on the vacuum to displace the field into a coherent configuration and utilizing the Baker-Campbell-Hausdorff (BCH) identity leads us to the fundamental relationship between coherent states and the ladder operators used throughout the following examples:

$$\begin{aligned} &a_{(+)}(k)\left|\alpha^{+}\right\rangle = \alpha(k)\left|\alpha^{+}\right\rangle \\ &a_{(-)}(k)\left|\alpha^{+}\right\rangle = 0 \\ &\left\langle\alpha^{+}\right| a^{\dagger}_{(+)}(k) = \left\langle\alpha^{+}\right| \alpha^{*}(k) \\ &\left\langle\alpha^{+}\right| a^{\dagger}_{(-)}(k) = 0 \end{aligned} \tag{3.4}$$

In the following section, we will use coherent states to obtain a classical electromagnetic wave from a quantum scattering amplitude. The following derivations in Sec. 3.3 was developed by Andrea Cristofoli, Riccardo Gonzo, David A. Kosower, and Donal O'Connell in [5]. Here we show more steps of their derivation and then later apply this process to the graviton field operator in Sec. 4.3, which is not shown in [5]. There are two derivations in this section. The first one is applying the coherent states to the electromagnetic field, $\mathbb{A}_\mu(x)$, at a point x in space. The second derivation is the factorization of expectation values of the EM fields at a point x and a point y in space. This derivation ensures that photon from one point (x) to another point (y) should be independent of each other. This gives us the classical behavior since it shows the path between the two points and is fully deterministic.

3.3 Classical Electromagnetic Fields from Coherent States

3.3.1 Classical EM Field From Coherent States

We are assuming the wave already exists in the state $|\alpha^+\rangle$ and proving it behaves classically (it factorizes and has no variance). Diving into the first derivation, we apply the coherent states ($|\alpha^+\rangle$), defined in Sec. 3.2, to the Electromagnetic Field $\mathbb{A}_\mu(x)$.

$$\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle = \left\langle \alpha^+ \left| \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(k) \left[a_{(\eta)}(k)\varepsilon^{(\eta)*}_\mu(k) e^{-ikx/\hbar} + a^\dagger_{(\eta)}(k)\varepsilon^{(\eta)}_\mu(k) e^{+ikx/\hbar}\right] \right| \alpha^+ \right\rangle \tag{3.5}$$

We distribute the coherent states. They are now being applied to the annihilation and creation operators only as they are operators, while the polarization and momentum are scalar and can be factored out.

$$= \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(k) \left[\varepsilon^{(\eta)*}_\mu(k) \langle\alpha^+|a_{(\eta)}|\alpha^+\rangle e^{-ikx/\hbar} + \langle\alpha^+|a^\dagger_{(\eta)}(k)|\alpha^+\rangle \varepsilon^{(\eta)}_\mu(k) e^{+ikx/\hbar}\right] \tag{3.6}$$

If you recall the relations from Eqn. 3.4, we can rewrite $a_{(\eta)}|\alpha^+\rangle$ as $\alpha(k)|\alpha^+\rangle$ and $\langle\alpha^+|a^\dagger_{(\eta)}(k)$ as $\langle\alpha^+|\alpha^*$

$$= \frac{1}{\sqrt{\hbar}} \int d\Phi(k) \left[\varepsilon^{+*}_\mu(k) \underbrace{\langle\alpha^+|\alpha(k)|\alpha^+\rangle}_{\langle\alpha^+|\alpha^+\rangle = 1} e^{-ikx/\hbar} + \langle\alpha^+|\alpha^*(k)|\alpha^+\rangle \varepsilon^{+}_\mu(k) e^{+ikx/\hbar} \right] \tag{3.7}$$

We can factor the waveshape function ($\alpha(k)$) out and remain with just the states applied to each other which is 1.

$$\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle = \frac{1}{\sqrt{\hbar}}\int d\Phi(k)\left[\alpha(k)\varepsilon_\mu^{+*}(k)e^{-ikx/\hbar} + \alpha^*(k)\varepsilon_\mu^{+}(k)e^{+ikx/\hbar}\right] \tag{3.8}$$

We are almost done. We must remember to rescale the momentum to the wavenumber. This is crucial because momentum is mainly used for point-like particles. However, since an EM wave is not a point-like particle we must rescale k so that we use the wavenumber since it measures the wavelength and allows us to keep the wave structure of the photon. This will be a crucial to obtain the classical limit since without the rescaling, as we take $\hbar$ to zero it will diverge. Since the wave nature is described by the wavenumber, when we take this limit, it will converge to classical behavior in a way momentum alone does not. In principle, quantum theory should reduce to a classical theory once $\hbar$ goes to 0 since it is already small which is why the world looks very classical to us. The fact that $\hbar$ is nonzero means nature is truly quantum mechanical.

Here, we rescale to wavenumber given by $\bar{k} = k/\hbar$ and $\alpha \sim \hbar^{-3/2}\bar{\alpha}(\bar{k})$ and the $\hbar$ cancels out. We are now left with the classical EM field.

$$A_{cl\,\mu}(x) \equiv \int d\Phi(k)\left[\overline{\alpha}(\bar{k})\varepsilon_\mu^{+*}(\bar{k})e^{-i\bar{k}x} + \overline{\alpha}^*(\bar{k})\varepsilon_\mu^{+}(\bar{k})e^{+i\bar{k}x}\right] \tag{3.9}$$

3.3.2 Factorization of Expectation EM Field

Although we have obtained the classical EM field this does not ensure that it behaves classically. We must require factorization of expectation values which is to make sure the photon fields from one point in space to the other are uncorrelated (i.e. independent). Classical predictions are fully deterministic. There should be zero variance and in our case zero covariance. In other words, there should be zero quantum uncertainty in a classical theory. Checking for this independence indicates classical behavior because it will mean the field's behavior is predictable and implicitly shows the path of the waveform extending between the two points in space (x and y) which reflects a classical like determinism.

In short, to ensure classical behavior the following relation must be met.

$$\left\langle\alpha^+\right|\mathbb{A}_\mu(x)\mathbb{A}_\nu(y)\left|\alpha^+\right\rangle \simeq \left\langle\alpha^+\right|\mathbb{A}_\mu(x)\left|\alpha^+\right\rangle\left\langle\alpha^+\right|\mathbb{A}_\nu(y)\mid\alpha^+\rangle \tag{3.10}$$

To show this is true, lets apply coherent states to the multiplication of the EM field at point x and EM field at point y.

$$\begin{aligned}\left\langle\alpha^+\right|\mathbb{A}_\mu(x)\mathbb{A}_\nu(y)\left|\alpha^+\right\rangle = \Bigg\langle\alpha^+\Bigg|&\underbrace{\left(\frac{1}{\sqrt{\hbar}}\sum_\eta\int d\Phi(k)\left[a_{(\eta)}(k)\varepsilon_\mu^{(\eta)*}(k)e^{-ikx/\hbar} + a^\dagger_{(\eta)}(k)\varepsilon_\mu^{(\eta)}(k)e^{+ikx/\hbar}\right]\right)}_{\mathbb{A}_\mu(x)}\\ &\times\underbrace{\left(\frac{1}{\sqrt{\hbar}}\sum_{\eta'}\int d\Phi(k')\left[a_{(\eta')}(k')\varepsilon_\nu^{(\eta')*}(k')e^{-ik'y/\hbar} + a^\dagger_{(\eta')}(k')\varepsilon_\nu^{(\eta')}(k')e^{+ik'y/\hbar}\right]\right)}_{\mathbb{A}_\nu(y)}\Bigg|\alpha^+\Bigg\rangle\end{aligned} \tag{3.11}$$

If you notice, there are two sums and two integrals. If you recall Eqn. 3.1, η is the helicity, k is the momentum, and ε is the polarization. Here we denote those symbols for the EM field at the x point ($\mathbb{A}_\mu(x)$) and for the EM field at the y point, we denote the symbols with an apostrophe(η',k',ε').

Now lets expand this by multiplying out the EM fields.

$$\left\langle \alpha^+ \middle| \frac{1}{\hbar} \sum_{\eta,\eta'} \int d\Phi(k) \int d\Phi(k') \Bigg[\underbrace{\left(a_{(\eta)}(k)\varepsilon_\mu^{(\eta)*}(k)\right) \left(a_{(\eta')}(k')\varepsilon_\nu^{(\eta')*}(k')\right) e^{-\frac{i}{\hbar}(kx+k'y)}}_{0} \right.$$
$$+ \left(a_{(\eta)}(k)\varepsilon_\mu^{(\eta)*}(k)\right) \left(a^\dagger_{(\eta')}(k')\varepsilon_\nu^{(\eta')}(k')\right) e^{-\frac{i}{\hbar}(kx-k'y)} + \left(a^\dagger_{(\eta)}(k)\varepsilon_\mu^{(\eta)}(k)\right) \left(a_{(\eta')}(k')\varepsilon_\nu^{(\eta')*}(k')\right) e^{\frac{i}{\hbar}(kx-k'y)}$$
$$\left. + \underbrace{\left(a^\dagger_{(\eta)}(k)\varepsilon_\mu^{(\eta)}(k)\right) \left(a^\dagger_{(\eta')}(k')\varepsilon_\nu^{(\eta')}(k')\right) e^{\frac{i}{\hbar}(kx+k'y)}}_{0} \Bigg] \middle| \alpha^+ \right\rangle$$

The first and last term go to 0 due to anti-commutation relations where $[a^\dagger, a^\dagger] = 0$ or $[a, a] = 0$ so $a^{\dagger 2} = 0$ and $a^2 = 0$.

With the remaining terms, lets rewrite the right hand side using the commutation relations.

$$\left\langle \alpha^\dagger \middle| \frac{1}{\hbar} \sum_{\eta,\eta'} \int d\Phi(k) \int d\Phi(k') \Bigg(a^\dagger_\eta(k) a_{(\eta')}(k') \varepsilon_\mu^{(\eta)}(k) \varepsilon_\nu^{(\eta')*}(k') e^{\frac{i}{\hbar}(kx-k'y)} \right.$$
$$\left. + \Bigg(\underbrace{[a_\eta(k), a^\dagger_{\eta'}(k')]}_{\hat{\delta}_\Phi(k-k')} + a^\dagger_{\eta'}(k') a_\eta(k) \Bigg) \varepsilon_\mu^{(\eta)*}(k) \varepsilon_\nu^{(\eta')}(k') e^{-\frac{i}{\hbar}(kx-k'y)} \Bigg) \middle| \alpha^\dagger \right\rangle \tag{3.12}$$

Now, lets distribute and apply the coherent states, once again using the relations from Eqn. 3.4.

$$\left\langle \alpha^+ \right| \mathbb{A}_\mu(x) \mathbb{A}_\nu(y) \left| \alpha^+ \right\rangle = \frac{1}{\hbar} \sum_\eta \sum_{\eta'} \int d\Phi(k) \int d\Phi(k') \Bigg[\Bigg(\hat{\delta}_\Phi(k-k') + \underbrace{\langle \alpha^+ | a^\dagger_{\eta'}(k')}_{\langle \alpha^+ | \alpha^*(k')} \underbrace{a_\eta(k) | \alpha^+ \rangle}_{\alpha(k) | \alpha^+ \rangle} \Bigg) \varepsilon_\mu^{\eta*}(k) \varepsilon_\nu^{\eta'}(k') e^{-\frac{i}{\hbar}(kx-k'y)}$$
$$+ \underbrace{\langle \alpha^+ | a^\dagger_\eta(k)}_{\langle \alpha^+ | \alpha^*(k)} \underbrace{a_{\eta'(k') | \alpha^+ \rangle}}_{\alpha(k') | \alpha^+ \rangle} \varepsilon_\mu^\eta(k) \varepsilon_\nu^{\eta' *}(k') e^{\frac{i}{\hbar}(kx-k'y)} \Bigg] \tag{3.13}$$

Simplifying and distributing the sums and integrals on the outside:

$$= \frac{1}{\hbar} \int d\Phi(k) \underbrace{\int d\Phi(k') \hat{\delta}_\Phi(k-k')}_{k=k', \hat{\delta}=1} \varepsilon_\mu^{+*}(k) \varepsilon_\nu^+(k') e^{-\frac{i}{\hbar}(kx-k'y)}$$
$$+ \frac{1}{\hbar} \int d\Phi(k) \int d\Phi(k') \alpha^*(k') \alpha(k) \varepsilon_\mu^{+*}(k) \varepsilon_\nu^+(k') e^{-\frac{i}{\hbar}(kx-k'y)}$$
$$+ \frac{1}{\hbar} \int d\Phi(k) \int d\Phi(k') \alpha^*(k) \alpha(k') \varepsilon_\mu^+(k) \varepsilon_\nu^{+*}(k') e^{\frac{i}{\hbar}(kx-k'y)} \tag{3.14}$$

If we take a look at the last two lines, we can employ a basic calculus technique. Keeping in mind the terms we crossed out to 0 earlier we get the following perfect square factorization.

$$= \frac{1}{\hbar}\int d\Phi(k) \underbrace{\sum_\eta \varepsilon_\mu^{\eta *}\varepsilon_\nu^\eta}_{\text{light cone gauge}} \; e^{-\frac{ik}{\hbar}(x-y)} + \overbrace{\left(\frac{1}{\sqrt{\hbar}}\int d\Phi(k)\left[\alpha(k)\varepsilon_\mu^{+*}(k)e^{-\frac{ikx}{\hbar}} + \alpha^*(k)\varepsilon_\mu^{+}(k)e^{\frac{ikx}{\hbar}}\right]\right)}^{\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle} \tag{3.15}$$

$$\times \underbrace{\left(\frac{1}{\sqrt{\hbar}}\int d\Phi(k')\left[\alpha^*(k')\varepsilon_\nu^{+}(k')e^{\frac{ik'y}{\hbar}} + \alpha(k')\varepsilon_\nu^{+*}(k')e^{-\frac{ik'y}{\hbar}}\right]\right)}_{\langle\alpha^+|\mathbb{A}_\nu(y)|\alpha^+\rangle} \tag{3.16}$$

To simplify the expectation value of the field product and ensure the non-classical terms go to 0 when $\hbar \to 0$, we must impose a gauge condition that isolates the physical degrees of freedom. The gauge that was chosen in [5] was the light-cone gauge. Since photons possess only two physical transverse polarizations, the light-cone gauge ensures that the polarization sum $\sum_\eta \epsilon_\mu^{\eta *}\epsilon_\nu^\eta$ accounts only for these physical states, effectively removing gauge-dependent terms that do not contribute to the observable radiation. Under this gauge, the expectation value of the field product is:

$$\langle\alpha^+|\mathbb{A}_\mu(x)\mathbb{A}_\nu(y)|\alpha^+\rangle = \frac{1}{\hbar}\int \hat{d}\Phi(k)\left[\eta_{\mu\nu} - \frac{k_\mu q_\nu + k_\nu q_\mu}{k\cdot q + i\delta}\right]e^{-ik(x-y)/\hbar} + \langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle\langle\alpha^+|\mathbb{A}_\nu(y)|\alpha^+\rangle \tag{3.17}$$

The term in the brackets represents the numerator of the light-cone gauge propagator. Unlike the Feynman propagator, which contains unphysical degrees of freedom, this numerator keeps only the physical transverse states. The presence of the k terms in the numerator is vital for the classical limit, as it ensures the correct cancellation of $\hbar$ factors when moving to the wavenumber representation.

Before taking the classical limit, we must rescale the momentum to the wavenumber $\bar{k} = k/\hbar$. Applying this scaling we obtain:

$$\langle\alpha|\mathbb{A}_\mu(x)\mathbb{A}_\nu(y)|\alpha\rangle = \hbar\int \hat{d}\Phi(\bar{k})\left[\eta_{\mu\nu} - \frac{\bar{k}_\mu q_\nu + \bar{k}_\nu q_\mu}{\bar{k}\cdot q + i\delta}\right]e^{-i\bar{k}(x-y)} + \langle\alpha|\mathbb{A}_\mu(x)|\alpha\rangle\langle\alpha|\mathbb{A}_\nu(y)|\alpha\rangle \tag{3.18}$$

Finally, by taking the limit $\hbar \to 0$, the first term becomes negligible and we recover the classical factorization of the fields as seen in Eq. 3.17.

3.4 Discussion

As demonstrated in the preceding sections, the framework of quantum mechanics can successfully reproduce classical deterministic wave behavior. By evaluating the electromagnetic field operator within a coherent state and taking the classical limit ($\hbar \to 0$), the quantum fluctuations are shown to vanish entirely. A critical element of this derivation was the use of the light-cone gauge, which isolated the physical degrees of freedom and allowed the expectation values of the field to factorize perfectly.

While this provides a robust mathematical foundation for electrodynamics, the primary objective of this research is to model gravitational radiation. Transitioning from the spin-1 photons of gauge theory to the spin-2 gravitons of General Relativity introduces some complexity. Attempting to calculate gravitational scattering amplitudes directly using standard Feynman diagram techniques is a headache due to self-interacting nature of gravity. The limitation of BHPT and standard Feynman diagram techniques for gravity directly motivates the exploration of the Double Copy framework and the S-matrix formalism in the following chapter.

4 Connections Between Gravity and Quantum Mechanics

4.1 Double Copy

Historically, General Relativity taught us to view gravity as the geometry of the universe, as the curving of the spacetime fabric. However, to reconcile this with the rest of physics, we must view gravity as a fundamental quantum field. In this framework, the curving we observe is not a static background, but a dynamic, mathematical description of graviton excitations. By treating the gravitational field as a bunch of particles, we can utilize the tools of scattering amplitudes to reconstruct the classical reality of black hole mergers and the gravitational waves they emit from the bottom up.

The Double Copy is a framework in quantum gravity that proposes a mathematical duality between gauge theories and gravity [2]. It demonstrates that scattering amplitudes in a gravitational theory can be constructed directly from a non-abelian gauge theory by replacing the color factor with an additional kinematic factor. Since the amplitude now contains two kinematic factors instead of one, it is referred to as a double copy. This relationship has simplified the calculation of complex gravitational scattering amplitudes. Today, this concept is being leveraged to calculate the physical shape of the gravitational waves emitted by two merging black holes, an application we explore in the derivation in Sec. 4.2.

For this duality to hold (specifically via the BCJ formulation [2]), the kinematic numerators (n) must be chosen such that they obey the exact same algebraic relations as the color factors, such as $n_s + n_t + n_u = 0$. This parallel structure is known as color-kinematics duality, rendering the kinematic numerator mathematically interchangeable with the color factor. With these kinematic numerators satisfying the condition, we can define a new tree-level amplitude corresponding to gravity ($\mathcal{M}$) by simply replacing the color factor (c) with a second kinematic numerator:

$$A_n^L = \sum_I \frac{n_I^L c_I^L}{\prod_{j\in I} p_j^2}, \quad A_n^R = \sum_I \frac{n_I^R c_I^R}{\prod_{j\in I} p_j^2} \quad \rightarrow \quad \mathcal{M}_n^{L\otimes R} = \sum_I \frac{n_I^L n_I^R}{\prod_{j\in I} p_j^2} \tag{4.1}$$

However, a scattering amplitude ($\mathcal{M}$) is just a probability amplitude for a single microscopic interaction, not a macroscopic physical wave you can measure with LIGO. To obtain the observable wave, we must evaluate the expectation value of a physical quantum operator, such as the Riemann curvature tensor, in the final state of a scattering event. In the following sections, we utilize the KMOC formalism[15], which provides the mathematical tools to extract these classical observables directly by S-matrix evolved states to the operator. The distinction between evaluating the field using coherent states and evaluating an operator on S-matrix evolved states is that coherent states provide the framework for the classical behavior of a wave that already exists, whereas the S-matrix allows us to calculate the creation of a wave from a physical scattering event.

4.2 Gravitational Spectrum using Riemann Curvature Tensor

To calculate the creation of the observable outgoing wave, we evaluate the expectation value of the Riemann curvature tensor in the final state of a scattering process.

First, we define the initial state of the scattering event. The incoming two-particle state is constructed using wave packet profiles (ϕ) and the impact parameter b:

$$|\psi\rangle_{\text{in}} = \int \mathrm{d}\Phi(p_1, p_2)\phi_b(p_1, p_2)|p_1, p_2\rangle_{\text{in}} \tag{4.2}$$

The corresponding bra state is:

$$\langle\psi| = \langle p_1', p_2'| \int \mathrm{d}\Phi(p_1', p_2')\phi_b^*(p_1', p_2') \tag{4.3}$$

Next, we introduce the operator for the observable radiation. The Riemann curvature operator in momentum space is defined as:

$$\mathbb{R}_{\mu\nu\rho\sigma}(x) = \frac{\kappa}{2} 2\,\mathrm{Re} \sum_{\eta=\pm} \int \mathrm{d}\Phi(k) k_{[\mu}\varepsilon_{\nu]}^{(-\eta)}(k) k_{[\rho}\varepsilon_{\sigma]}^{(-\eta)}(k)\mathrm{e}^{-\mathrm{i}k\cdot x} a_\eta(k) \tag{4.4}$$

To find the expected curvature after the scattering event, we evaluate $\langle\psi|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(x) S|\psi\rangle$. Since the S-matrix acts directly on the field operators, we first evaluate how it transforms the annihilation operator $a_\eta(k)$ from Eqn. 4.4. Using $S = 1 + iT$:

$$\begin{aligned}\langle\psi|S^\dagger a_\eta(k)S|\psi\rangle &= \langle\psi|(1-iT^\dagger)a_\eta(k)(1+iT)|\psi\rangle \\ &= \langle\psi|a_\eta(k)-iT^\dagger a_\eta(k)(1+iT)|\psi\rangle \\ &= \langle\psi|a_\eta(k)+ia_\eta(k)T-iT^\dagger a_\eta(k)+T^\dagger a_\eta(k)T|\psi\rangle \\ &= \underbrace{\langle\psi|a_\eta(k)|\psi\rangle}_{0}+i\langle\psi|a_\eta(k)T|\psi\rangle - i\underbrace{\langle\psi|T^\dagger a_\eta(k)|\psi\rangle}_{0}+\langle\psi|T^\dagger a_\eta(k)T|\psi\rangle \\ &= i\langle\psi|a_\eta(k)T|\psi\rangle+\langle\psi|T^\dagger a_\eta(k)T|\psi\rangle \end{aligned} \tag{4.5}$$

The terms $\langle\psi|a_\eta(k)|\psi\rangle$ and $\langle\psi|T^\dagger a_\eta(k)|\psi\rangle$ vanish because $a_\eta(k)|\psi\rangle = 0$, reflecting the physical fact that there is no incoming radiation in the initial state. The surviving term $i\langle\psi|a_\eta(k)T|\psi\rangle$ is equivalent to $(a_\eta^\dagger(k)|\psi\rangle)^\dagger$, which represents the creation of a soft graviton in the final state, leading to a 5-point amplitude.

Substituting the evaluated S-matrix action (Eqn. 4.5) and the wave packet states back into the expectation value for the Riemann curvature operator, we obtain the expanded form:

$$\begin{aligned}\langle\psi|S^\dagger R_{\mu\nu\rho\sigma}(x)S|\psi\rangle = \frac{\kappa}{2}2\mathrm{Re}\sum_{\eta=\pm}\int \mathrm{d}\Phi(k,p_1',p_2',p_1,p_2)\, k_{[\mu}\epsilon_{\nu]}^{(-\eta)}(k)k_{[\rho}\epsilon_{\sigma]}^{(-\eta)}(k)\, e^{-ik\cdot x}\phi_b^*(p_1',p_2')\phi_b(p_1,p_2) \\ \times\left[i\underbrace{\langle p_1',p_2',k^\eta|T|p_1,p_2\rangle}_{\text{5 point amplitude}}+\underbrace{\langle p_1',p_2'|T^\dagger a_\eta(k)T|p_1,p_2\rangle}_{\text{Product of amplitudes (summing over possibilities)}}\right]\end{aligned} \tag{4.6}$$

Here, k^η is the momentum of the emitted soft graviton. At this stage, we can express the T-matrix element as a standard scattering amplitude. The scattering amplitudes are the matrix elements of the T-operator between plane-wave states[5]:

$$\langle p_1'\dots p_m'|\,T\,|p_1\dots p_n\rangle = \mathcal{M}\left(p_1\dots p_n\to p_1'\dots p_m'\right)\hat{\delta}^{(4)}\left(p_1+\dots+p_n-p_1'-\dots-p_m'\right) \tag{4.7}$$

We rewrite the expectation value by substituting the amplitude $\mathcal{M}$:

$$\begin{aligned}\langle\psi|S^\dagger\mathbb{R}_{\mu\nu\rho\sigma}(x)S|\psi\rangle = \kappa\,\mathrm{Re}\sum_{\eta=\pm}\int \mathrm{d}\Phi(k,p_1',p_2',p_1,p_2)\, k_{[\mu}\varepsilon_{\nu]}^{(-\eta)}(k)k_{[\rho}\varepsilon_{\sigma]}^{(-\eta)}(k)\,\mathrm{e}^{-ik\cdot x}\phi_b^*(p_1',p_2')\phi_b(p_1,p_2) \\ \times i\mathcal{M}(p_1p_2\to p_1'p_2'k^\eta)\hat{\delta}^{(4)}(p_1'+p_2'-p_1-p_2-k)\end{aligned} \tag{4.8}$$

To evaluate this integral, we relate the final momenta p_1', p_2' and the radiated momentum k to the initial momenta p_1, p_2 via the momentum transfers q_1, q_2:

$$\begin{aligned}p_1' &= p_1 - q_1\\ p_2' &= p_2 - q_2\\ k &= q_1+q_2\end{aligned}$$

Where q_1, q_2 represent the momentum transferred from the particles to the gravitational radiation, and k is the total momentum radiated away. The integral is expanded and simplified by shifting the integration variables to the momentum transfers q_1 and q_2.

$$\int \mathrm{d}\Phi(k,p_1',p_2',p_1,p_2) = \int \mathrm{d}\Phi(k,p_1-q_1,p_2-q_2,p_1,p_2) \tag{4.9}$$

$$= \int \mathrm{d}\Phi(k,p_1,p_2)\int \mathrm{d}\Phi(p_1-q_1)\int \mathrm{d}\Phi(p_2-q_2) \tag{4.10}$$

$$= \int \mathrm{d}\Phi(k,p_1,p_2)\int \hat{d}^4q_1\hat{d}^4q_2\,\hat{\delta}\left((p_1-q_1)^2-m_1^2\right)\hat{\delta}\left((p_2-q_2)^2-m_2^2\right)\theta(p_1^0-q_1^0)\theta(p_2^0-q_2^0) \tag{4.11}$$

Using the on-shell condition ($p_i^2 = m_i^2$) and expanding the delta functions:

$$\begin{aligned}(p_1 - q_1)^2 - m_1^2 &= p_1^2 - 2p_1 \cdot q_1 + q_1^2 - m_1^2 \\ &= -2p_1 \cdot q_1 + q_1^2\end{aligned}$$

In the classical limit, where $q_i^0 \ll p_i^0$ (preventing pair production, no virtual particles), the step functions $\theta \approx 1$ is to ensure causality. The final simplified phase space becomes:

$$\int \mathrm{d}\Phi(k, p_1', p_2', p_1, p_2) = \int \mathrm{d}\Phi(k, p_1, p_2) \int \hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{\delta}(2p_1 \cdot q_1 - q_1^2)\hat{\delta}(2p_2 \cdot q_2 - q_2^2)$$

Substituting this back into the expectation we get:

$$\begin{aligned}\langle\psi|S^\dagger R_{\mu\nu\rho\sigma}(x) S|\psi\rangle &= \kappa\, \mathrm{Re} \sum_{\eta=\pm} \int \mathrm{d}\Phi(k, p_1, p_2) \int \hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{\delta}(2p_1 \cdot q_1 - q_1^2)\hat{\delta}(2p_2 \cdot q_2 - q_2^2) \\ &\times k_{[\mu}\epsilon_{\nu]}^{(-\eta)}(k) k_{[\rho}\epsilon_{\sigma]}^{(-\eta)}(k)\, \mathrm{e}^{-ik\cdot x} \phi_b^*(p_1 - q_1, p_2 - q_2)\phi_b(p_1, p_2) i\mathcal{A}(p_1 p_2 \to p_1' p_2' k^\eta)\hat{\delta}(q_1 + q_2 - k)\end{aligned} \tag{4.12}$$

The phase factors from the wave packet definitions are combined to pull out the impact parameter dependence, since in the classical regime the construction of the wave packet does not affect the waveform.

$$\begin{aligned}&\phi^*(p_1 - q_1)\phi^*(p_2 - q_2)\mathrm{e}^{i(p_1')\cdot b/\hbar}\phi(p_1)\phi(p_2)\mathrm{e}^{i(p_1)\cdot b/\hbar} \\ &\phi^*(p_1 - q_1)\phi^*(p_2 - q_2)\phi(p_1)\phi(p_2)\mathrm{e}^{iq_1\cdot b/\hbar}\end{aligned}$$

Now here is the final simplified expectation value for the gravitational radiation field.

$$\begin{aligned}\langle\psi|S^\dagger R_{\mu\nu\rho\sigma}(x) S|\psi\rangle = \kappa\, \mathrm{Re} \sum_{\eta=\pm} \int \mathrm{d}\Phi(k, p_1, p_2) \int \hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{\delta}(2p_1 \cdot q_1 - q_1^2)\hat{\delta}(2p_2 \cdot q_2 - q_2^2)\, k_{[\mu}\epsilon_{\nu]}^{(-\eta)}(k) k_{[\rho}\epsilon_{\sigma]}^{(-\eta)}(k) \\ \times \mathrm{e}^{-ik\cdot x}|\phi(p_1)\phi(p_2)|^2 \mathrm{e}^{iq_1\cdot b/\hbar} i\mathcal{M}(p_1 p_2 \to p_1' p_2' k^\eta)\hat{\delta}(q_1 + q_2 - k)\end{aligned} \tag{4.13}$$

Finally using the notation $\langle\langle f(p_1, p_2, \ldots)\rangle\rangle \equiv \int \mathrm{d}\Phi(p_1, p_2)\, v\, |\phi_1(p_1)|^2 |\phi_2(p_2)|^2 f(p_1, p_2, \ldots)$, where summing over all momentum states the probabilities ($|\phi_1(p_1)|^2 |\phi_2(p_2)|^2$) will be 1.

$$\begin{aligned}\langle\psi|S^\dagger R_{\mu\nu\rho\sigma} S|\psi\rangle = \kappa\ \mathrm{Re} \sum_{\eta=\pm} \Big\langle\Big\langle \int d\Phi(k)\hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{\delta}(2p_1 \cdot q_1)\hat{\delta}(2p_2 \cdot q_2)\hat{\delta}(q_1 + q_2 - k) \\ \times\ k_{[\mu}\varepsilon_{\nu]}^{(-\eta)}(k) k_{[\rho}\varepsilon_{\sigma]}^{(-\eta)}(k)\, e^{i(q_1\cdot b - k\cdot x)}\ \times i\mathcal{M}(p_1 p_2 \to p_1' p_2' k^\eta)\Big\rangle\Big\rangle\end{aligned} \tag{4.14}$$

Using the expectation, we can construct the current. The expectation value of the Riemann curvature operator $R_{\mu\nu\rho\sigma}$ is expressed in terms of the spectral density $\tilde{J}_{\mu\nu\rho\sigma}(k)$ as follows:

$$\langle\psi|S^\dagger R_{\mu\nu\rho\sigma} S|\psi\rangle = 2\,\mathrm{Re}\, i \int d\Phi(k)\tilde{J}_{\mu\nu\rho\sigma}(k)\mathrm{e}^{-ik\cdot x}$$

The current is derived by averaging over the wave packet profiles (denoted by $\langle\langle\ldots\rangle\rangle$) in the classical limit:

$$\begin{aligned}\tilde{J}_{\mu\nu\rho\sigma}(k) = -i\kappa \sum_{\eta=\pm} \Big\langle\Big\langle \int \hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{\delta}(2p_1 \cdot q_1)\hat{\delta}(2p_2 \cdot q_2)\hat{\delta}(q_1 + q_2 - k) \\ \times\ ik_{[\mu}\epsilon_{\nu]}^{-\eta}(k) k_{[\rho}\epsilon_{\sigma]}^{-\eta}(k)\mathrm{e}^{iq_1\cdot b}\mathcal{M}(p_1 p_2 \to p_1' p_2' k^\eta)\Big\rangle\Big\rangle\end{aligned}$$

With this current, we compute the integral over spatial components of the wave vector. We define the wave vector as $k = (\omega, \omega\hat{\mathbf{n}})$, where ω is the frequency and n is the direction of the wave.

$$\tilde{J}_{\mu\nu\rho\sigma}(\omega, \omega\mathbf{n}) = -i\kappa \sum_{\eta=\pm} \left\langle\!\!\left\langle \int \hat{d}^4 q_1 \hat{d}^4 q_2\, \hat{\delta}(2p_1 \cdot q_1)\hat{\delta}(2p_2 \cdot q_2)\hat{\delta}(q_1 + q_2 - \omega(1, \mathbf{n})) \right.\right.$$
$$\left.\left. \times\ i\omega^2 n_{[\mu}\epsilon^{-\eta}_{\nu]}(\omega, \omega\mathbf{n}) n_{[\rho}\epsilon^{-\eta}_{\sigma]}(\omega, \omega\mathbf{n}) e^{iq_1 \cdot b}\mathcal{M}(p_1 p_2 \to p_1' p_2' k^\eta) \right\rangle\!\!\right\rangle$$

We write the expectation in terms of the current:

$$\langle\psi|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(x)S|\psi\rangle = 2\text{Re}\ i \int d\Phi(k) e^{-i\omega x^0} e^{-i\omega\hat{\mathbf{n}}} \tilde{J}_{\mu\nu\rho\sigma}(\omega, \omega\hat{\mathbf{n}})$$

We start with the on-shell phase space measure and apply the scaling property of the delta function:

$$\int d\Phi(k) = \int \hat{d}^4 k \hat{\delta}(k^2)$$
$$= \int \hat{d}^4 k \hat{\delta}(\omega^2 - \omega^2 \mathbf{n}^2)$$

Applying the identity $\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$:

$$\hat{\delta}(\omega^2(1 - \mathbf{n}^2)) = \frac{\hat{\delta}(1 - \mathbf{n}^2)}{\omega^2}$$

Change to spherical coordinates, $\mathbf{n}$ is the vector representation of the angular direction of the outgoing wave. $\hat{d}^4 k \to \omega^2 \hat{d}\omega \hat{d}^3 n$:

$$= \int \omega^2 \hat{d}\omega \int \hat{d}^3 n \frac{\hat{\delta}(1 - \mathbf{n}^2)}{\omega^2} e^{-i\omega x^0} e^{i\omega \mathbf{n}\cdot\mathbf{x}} \tilde{J}_{\mu\nu\rho\sigma}(\omega, \omega\mathbf{n})$$

The ω^2 from the measure and the $1/\omega^2$ from the identity cancel:

$$\langle\psi|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(x)S|\psi\rangle = 2\text{Re}\ i \int \hat{d}\omega \int \hat{d}^3 n \hat{\delta}(1 - n^2) e^{-i\omega x^0} e^{i\omega\mathbf{n}\cdot\mathbf{x}} \tilde{J}_{\mu\nu\rho\sigma}(\omega, \omega\mathbf{n}) \tag{4.15}$$

Finally, writing the delta function in its integral representation to prepare for the Stationary Phase Approximation:

$$= 2\text{Re}\ i \int \hat{d}\omega \int \hat{d}^3 n \int d\lambda e^{i\lambda(1-n^2)} e^{-i\omega x^0} e^{i\omega\mathbf{n}\cdot\mathbf{x}} \tilde{J}_{\mu\nu\rho\sigma}(\omega, \omega\mathbf{n}) \tag{4.16}$$

Now integrating over n and λ: The general stationary phase approximation is

$$\int d^n z\, e^{if(z)} g(z) \simeq (2\pi)^{n/2} \sum_{z_0} e^{if(z_0)} g(z_0) \frac{1}{\sqrt{|\det H(z_0)|}} e^{i\frac{\pi}{4}\text{sign } H(z_0)} \tag{4.17}$$

With the condition that $f'(z_0) = 0$. In this specific case, the integration variables are the unit vector $\mathbf{n}$ and the parameter λ:

$$z = (\mathbf{n}, \lambda)$$

The phase function is defined as:

$$f(z) = \lambda(1 - \mathbf{n}^2) + \omega\mathbf{n}\cdot\mathbf{x} = \lambda - \lambda\mathbf{n}^2 + \omega\mathbf{n}\cdot\mathbf{x}$$

We take the partial derivatives with respect to our variables and set them to zero. The derivative with respect to $\mathbf{n}$ reads:

$$\frac{\partial f}{\partial \mathbf{n}} = -2\lambda\mathbf{n} + \omega\mathbf{x} = 0 \implies 2\lambda\mathbf{n} = \omega\mathbf{x}$$

Next, the derivative with respect to λ:

$$\frac{\partial f}{\partial \lambda} = 1 - \mathbf{n}^2 = 0 \implies \mathbf{n}^2 = 1$$

From the λ derivative, we know $\mathbf{n}$ must be a unit vector. Looking at the direction of $\mathbf{x}$, this implies $\mathbf{n}$ must point along the wave direction:

$$\mathbf{n} \cdot \mathbf{n} = \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \implies \mathbf{n} = \pm\frac{\mathbf{x}}{|\mathbf{x}|}$$

Substituting this back into the first condition allows us to find λ:

$$2\lambda\left(\pm\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \omega\mathbf{x} \implies \lambda = \pm\frac{\omega}{2}|\mathbf{x}|$$

The Hessian matrix H is the 4×4 matrix of second derivatives of the phase $f(\mathbf{n}, \lambda)$. Using the stationary points $z_0 = (\mathbf{n}, \lambda)$:

$$H = \begin{pmatrix} -2\lambda & 0 & 0 & -2n_x \\ 0 & -2\lambda & 0 & -2n_y \\ 0 & 0 & -2\lambda & -2n_z \\ -2n_x & -2n_y & -2n_z & 0 \end{pmatrix}$$

$$\det(H) = -2\lambda \begin{vmatrix} -2\lambda & 0 & -2n_y \\ 0 & -2\lambda & -2n_z \\ -2n_y & -2n_z & 0 \end{vmatrix} + 2n_x \begin{vmatrix} 0 & 0 & -2n_x \\ -2\lambda & 0 & -2n_y \\ 0 & -2\lambda & -2n_z \end{vmatrix}$$

$$\det(H) = (-2\lambda)(-2\lambda)(0 - 4n_z^2) + \cdots = -16\lambda^2 n_z^2$$

Substituting the stationary point value $\lambda^2 = \frac{\omega^2}{4}|\mathbf{x}|^2$ and using the fact that $|\mathbf{n}|^2 = 1$:

$$\det(H) = -16\frac{\omega^2|\mathbf{x}|^2}{4} = -4\omega^2|\mathbf{x}|^2 \tag{4.18}$$

Using the stationary points for the phase $f(z_0) = \pm\omega|\mathbf{x}|$ and the determinant result $\sqrt{|\det H|} = 2\omega|\mathbf{x}|$, we can evaluate the integral from Eqn. 4.17:

$$\int d^4z\, e^{if(z)} g(z) = 4\pi^2 \left[e^{i\omega|\mathbf{x}|} g(z_{0,ret}) \frac{1}{2\omega|\mathbf{x}|} e^{i\frac{\pi}{4}(-2)} + e^{-i\omega|\mathbf{x}|} g(z_{0,adv}) \frac{1}{2\omega|\mathbf{x}|} e^{i\frac{\pi}{4}(2)} \right] \tag{4.19}$$

Starting from the result of the stationary phase sum (including both retarded and advanced points):

$$\langle\psi|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(x) S|\psi\rangle = \mathrm{Re}\ \frac{1}{4\pi|\mathbf{x}|} \int_0^\infty \hat{d}\omega \left[e^{-i\omega(x^0-|\mathbf{x}|)} \frac{1}{\omega} \tilde{J}_{\mu\nu\rho\sigma}\left(\omega, \omega\frac{\mathbf{x}}{|\mathbf{x}|}\right) - e^{-i\omega(x^0+|\mathbf{x}|)} \frac{1}{\omega} \tilde{J}_{\mu\nu\rho\sigma}\left(\omega, -\omega\frac{\mathbf{x}}{|\mathbf{x}|}\right) \right] \tag{4.20}$$

We ignore the second term because it represents the advanced signal (radiation coming from infinity toward the source). We focus only on the signal after it reaches the observer:

$$\langle\psi|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma}(x) S|\psi\rangle = \mathrm{Re}\ \frac{1}{4\pi|\mathbf{x}|} \int_0^\infty \hat{d}\omega \left[e^{-i\omega(x^0-|\mathbf{x}|)} \frac{1}{\omega} \tilde{J}_{\mu\nu\rho\sigma}\left(\omega, \omega\frac{\mathbf{x}}{|\mathbf{x}|}\right) \right] \tag{4.21}$$

To write this as a standard Fourier integral from $-\infty$ to ∞, we use the reality condition (absorbing the 2Re by extending the limits):

$$= \frac{1}{8\pi|\mathbf{x}|\omega} \int_{-\infty}^{\infty} \hat{d}\omega e^{-i\omega(x^0-|\mathbf{x}|)} \tilde{J}_{\mu\nu\rho\sigma}\left(\omega, \omega\frac{\mathbf{x}}{|\mathbf{x}|}\right) \tag{4.22}$$

We now define the spectral waveform $f_{\mu\nu\rho\sigma}(\omega, \mathbf{n})$ as

$$f_{\mu\nu\rho\sigma}(\omega, \mathbf{n}) = \frac{\tilde{J}_{\mu\nu\rho\sigma}(\omega, \mathbf{n})}{8\pi\omega}$$

This allows us to write the final, compact expression for the Riemann curvature:

$$\langle\psi|S^\dagger \mathbb{R}_{\mu\nu\rho\sigma} S|\psi\rangle = \int \hat{d}\omega e^{-i\omega(x^0 - |\mathbf{x}|)} \frac{f_{\mu\nu\rho\sigma}(\omega, \mathbf{n})}{|\mathbf{x}|} \tag{4.23}$$

Plugging in the definition of the gravitational current , the spectral waveform for $\omega > 0$ is:

$$\begin{aligned} f_{\mu\nu\rho\sigma}(\omega, \mathbf{n}) = \frac{\kappa}{8\pi\omega} \sum_{\eta=\pm} \Big\langle\Big\langle \int & \hat{d}^4 q_1 \hat{d}^4 q_2 \, \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}(2p_2 \cdot q_2) \hat{\delta}(q_1 + q_2 - k) \\ & \times \; i k_{[\mu} \epsilon^{-\eta}_{\nu]}(k) k_{[\rho} \epsilon^{-\eta}_{\sigma]}(k) e^{iq_1 \cdot b} \mathcal{M}(p_1 p_2 \to p_1' p_2' k^\eta) \Big\rangle\Big\rangle \end{aligned} \tag{4.24}$$

As seen in Eqn. 4.24, the derived spectral waveform differs from the established result in [5] by an overall factor of $\frac{1}{\omega}$, and it seems reasonable for this to appear as it suppresses the high-energy modes of the spectrum.

The S-matrix calculation we just concluded shows the observable waveform. By evaluating the Riemann curvature tensor between the S-matrix evolved states, we directly calculate the radiation generated by a specific physical event (the scattering of the two initial masses).

On the other hand, the calculation in Sec. 4.3 applies coherent states directly to the graviton field operator to describe a classical gravitational wave that already exists. It successfully demonstrates that a specific quantum superposition of gravitons mirrors a classical wave, but it does not account for the source that emitted it.

Both calculations are necessary for a complete picture. The coherent state derivation proves the foundational concept that gravitons can form the classical waves detected by observatories like LIGO. Meanwhile, the S-matrix Riemann calculation provides the tools to generate those waves from microscopic interactions.

4.3 Classical Gravitational Field From Coherent States

In Section 3.3, it was established that applying the KMOC formalism to an EM field operator in a coherent state yields a classical, deterministic waveform. A natural extension is to apply this exact procedure to the graviton field operator. The motivation is to show whether a classical gravitational observable can be extracted directly from coherent states, which to my knowledge has not been explicitly worked out.

The derivation begins with the linearized graviton field operator $\hat{h}^{\mu\nu}(x)$ [9]. The phase space measure $\hat{d}\Phi(k) \equiv \frac{d^4k}{(2\pi)^4}\hat{\delta}(k^2)$ is utilized, explicitly restricting the integration to on-shell, massless gravitons.

$$\hat{h}^{\mu\nu}(x) = \int \frac{\hat{d}\Phi}{\sqrt{\hbar}} \sum_{\eta=1}^{2} \epsilon^{\mu\nu}_\eta(k) \left[\hat{a}_\eta(k) e^{i\frac{k\cdot x}{\hbar}} + \hat{a}^\dagger_\eta(k) e^{-i\frac{k\cdot x}{\hbar}}\right]$$

To determine the classical field, the expectation value of this operator is evaluated between coherent states $|\alpha^+\rangle$. As demonstrated previously, the annihilation operator acting on a coherent state returns the waveshape function $\alpha(k)$, and the creation operator acting to the left returns $\alpha^*(k)$.

$$\begin{aligned} \langle\alpha^+|\hat{h}^{\mu\nu}(x)|\alpha^+\rangle &= \langle\alpha^+| \int \frac{\hat{d}\Phi}{\sqrt{\hbar}} \sum_{\eta=1}^{2} \left(\epsilon^{\mu\nu}_\eta(k)\hat{a}_\eta(k) e^{i\frac{k\cdot x}{\hbar}} + \epsilon^{\mu\nu*}_\eta(k)\hat{a}^\dagger_\eta(k) e^{-i\frac{k\cdot x}{\hbar}}\right)|\alpha^+\rangle \\ &= \int \frac{\hat{d}\Phi}{\sqrt{\hbar}} \left(\epsilon^{\mu\nu}_\eta(k)\langle\alpha^+|\hat{a}_\eta(k)|\alpha^+\rangle e^{i\frac{k\cdot x}{\hbar}} + \epsilon^{\mu\nu*}_\eta(k)\langle\alpha^+|\hat{a}^\dagger_\eta(k)|\alpha^+\rangle e^{-i\frac{k\cdot x}{\hbar}}\right) \\ &= \int \hat{d}\Phi(\bar{k}) \left[\epsilon^{\mu\nu}_\eta(k)\alpha(k) e^{i\frac{k\cdot x}{\hbar}} + \epsilon^{\mu\nu*}_\eta(k)\alpha^*(k) e^{-i\frac{k\cdot x}{\hbar}}\right] \end{aligned}$$

To transition to the classical limit, the KMOC scaling rules are applied for the classical wavenumber $\bar{k}$ and the coherent state $\bar{\alpha}$:

$$\bar{k} = \frac{k}{\hbar} \to k = \hbar\bar{k} \qquad \bar{\alpha} = \hbar^{3/2}\alpha \to \alpha = \hbar^{-3/2}\bar{\alpha}$$

Substituting these scalings into the integral allows for the $\hbar$s to cancel leaving a classical field $h^{\mu\nu}_{\text{classical}}$:

$$= \int \hat{d}\Phi \left[\epsilon^{\mu\nu}_{\eta}(\hbar\bar{k})\hbar^{-3/2}\bar{\alpha}(\bar{k})e^{i\bar{k}\cdot x} + \epsilon^{\mu\nu *}_{\eta}(\hbar\bar{k})\hbar^{-3/2}\bar{\alpha}^{*}(\bar{k})e^{-i\bar{k}\cdot x}\right]$$
$$= \int \hat{d}\Phi(\bar{k}) \left[\epsilon^{\mu\nu}_{\eta}(\bar{k})\bar{\alpha}(\bar{k})e^{i\bar{k}\cdot x} + \epsilon^{\mu\nu *}_{\eta}(\bar{k})\bar{\alpha}^{*}(\bar{k})e^{-i\bar{k}\cdot x}\right] = h^{\mu\nu}_{\text{classical}}$$

While a classical field is identified, a valid classical theory must be deterministic, exhibiting zero quantum variance. To verify this, the expectation value of the product of two graviton field operators at different spacetime points, x and y, must be considered. The following factorization is required to hold:

$$\langle\alpha^{+}|\hat{h}_{\mu\nu}(x)\hat{h}_{\rho\sigma}(y)|\alpha^{+}\rangle \simeq \langle\alpha^{+}|\hat{h}_{\mu\nu}(x)|\alpha^{+}\rangle\langle\alpha^{+}|\hat{h}_{\rho\sigma}(y)|\alpha^{+}\rangle \tag{4.25}$$

Expanding the field operators:

$$\langle\alpha^{+}|\hat{h}_{\mu\nu}(x)\hat{h}_{\rho\sigma}(y)|\alpha^{+}\rangle = \langle\alpha^{+}| \left(\frac{1}{\sqrt{\hbar}}\sum_{\eta}\int \hat{d}\Phi(k) \left[\epsilon^{\mu\nu}_{\eta}(k)\hat{a}_{\eta}(k)e^{ik\cdot x/\hbar} + \epsilon^{\mu\nu *}_{\eta}(k)\hat{a}^{\dagger}_{\eta}(k)e^{-ik\cdot x/\hbar}\right]\right)$$
$$\times \left(\frac{1}{\sqrt{\hbar}}\sum_{\eta'}\int \hat{d}\Phi(k') \left[\epsilon^{\rho\sigma}_{\eta'}(k')\hat{a}_{\eta'}(k')e^{ik'\cdot y/\hbar} + \epsilon^{\rho\sigma *}_{\eta'}(k')\hat{a}^{\dagger}_{\eta'}(k')e^{-ik'\cdot y/\hbar}\right]\right)|\alpha^{+}\rangle \tag{4.26}$$

Expanding the product and using the property of coherent states $\hat{a}_{\eta}(k)|\alpha^{+}\rangle = \alpha_{\eta}(k)|\alpha^{+}\rangle$, we focus on the non-vanishing terms. We use the commutator relation to reorder the creation and annihilation operators:

$$\hat{a}_{\eta}(k)\hat{a}^{\dagger}_{\eta'}(k') = [\hat{a}_{\eta}(k), \hat{a}^{\dagger}_{\eta'}(k')] + \hat{a}^{\dagger}_{\eta'}(k')\hat{a}_{\eta}(k)$$

where the commutator gives the on-shell delta function $\hat{\delta}_{\eta\eta'}\hat{\delta}(k-k')$.

The expectation value then separates into a quantum term (from the commutator) and the classical field product:

$$= \frac{1}{\hbar}\int \hat{d}\Phi(k)\sum_{\eta}\epsilon^{\mu\nu}_{\eta}(k)\epsilon^{\rho\sigma *}_{\eta}(k)$$
$$+ \left[\int \hat{d}\Phi(k)\left(\alpha(k)\epsilon^{\mu\nu}_{+}(k)e^{ik\cdot x/\hbar} + \alpha^{*}(k)\epsilon^{\mu\nu *}_{+}(k)e^{-ik\cdot x/\hbar}\right)\right]$$
$$\times \left[\int \hat{d}\Phi(k')\left(\alpha(k')\epsilon^{\rho\sigma}_{+}(k')e^{ik'\cdot y/\hbar} + \alpha^{*}(k')\epsilon^{\rho\sigma *}_{+}(k')e^{-ik'\cdot y/\hbar}\right)\right]$$

The final line identifies the classical field components:

$$\langle\alpha^{+}|\hat{h}_{\mu\nu}(x)\hat{h}_{\rho\sigma}(y)|\alpha^{+}\rangle = \frac{1}{\hbar}\int \hat{d}\Phi(k)\sum_{\eta}\epsilon^{\mu\nu}_{\eta}(k)\epsilon^{\rho\sigma *}_{\eta}(k) + h^{cl}_{\mu\nu}(x)h^{cl}_{\rho\sigma}(y) \tag{4.27}$$

For gravity, the standard textbook choice to isolate physical polarizations is the Transverse-Traceless (TT) gauge. In classical General Relativity, this gauge is ideal because it removes unphysical freedoms, leaving only the two propagating degrees of freedom (the plus and cross polarizations) for a gravitational wave.

However, applying this classical gauge to a quantum scaling limit reveals a dimensional mismatch. The polarization sum in the TT gauge relies on projection operators constructed from normalized momentum vectors (e.g., $k_i k_j/|\vec{k}|^2$). Since these combinations are dimensionally scale-invariant, applying the KMOC classical scaling $k = \hbar\bar{k}$, as we did for the electromagnetic radiation in 3.3, causes the $\hbar$ factors to cancel out entirely. Thus, the TT gauge fails to provide the overall $\hbar$ factor in the numerator required to suppress the $1/\hbar$ divergence. When the classical limit $\hbar \to 0$ is taken, the quantum fluctuation term survives, and the expectation value fails to cleanly factorize.

This presents the question of how must the gauge be chosen to successfully extract a classical gravitational wave directly from the field operator. One potential way is to investigate whether there is a gravitational analogue to the electromagnetic light-cone gauge that isolates the physical states while possessing the necessary dimensional scaling behavior, work that discusses light cone gauge for gravity is [18]. Alternatively, it may be possible to construct the necessary gravitational propagator directly via the Double Copy of the electromagnetic light-cone gauge itself.

4.4 Discussion

A central question in this research is how the predictions of the Double Copy framework compare to those of classical Black Hole Perturbation Theory (BHPT), and whether this quantum approach can ultimately provide an approach to calculate the waveform that observatories like LIGO can detect. Both frameworks are attempting to describe the exact same physical reality. BHPT starts with a curved classical geometry (such as the Kerr metric) and solving for small perturbations to the metric. In contrast, the Double Copy builds the gravitational field from the scattering of massless spin-2 gravitons in flat space. If the KMOC formalism and Double Copy techniques are fully developed, they should not only reproduce the classical waveforms predicted by BHPT but do so much more efficiently, avoiding the non-linear complexities of the Einstein Field Equations.

However, the current progress of extracting observable gravitational waves from scattering amplitudes is heavily skewed toward specific phases of the merger. So far, the literature has seen success applying the Double Copy to the inspiral phase of binary systems, specifically through Post-Minkowskian expansions that calculate the conservative dynamics and scattering angles of two approaching black holes [3]. Calculating the radiation emitted during the inspiral and the initial merger event has proven to be an ideal scenario for S-matrix calculations, where the initial state consists of two distinct, incoming particles ($|p_1, p_2\rangle$).

The ringdown phase remains a largely uncharted area for the Double Copy and scattering amplitude techniques. The fundamental difficulty lies in defining the quantum states. The ringdown is not the collision of two particles, but the settling of a single, excited, bound state. Defining the initial quantum state of a newly merged, oscillating Kerr black hole so that it can be processed by an S-matrix or evaluated using coherent states is complex due to the dynamical spacetime. While recent theoretical work such as explorations into the classical limit of gravitational radiation from amplitudes [5, 4] provides the mathematical tools for extracting wave observables, explicit calculations of a complete, time-domain ringdown waveform using the Double Copy and amplitude techniques are not present in the literature.

This gap in the research shows both the difficulty and the necessity of this work. While finding the correct gauge and propagator to seamlessly transition from a quantum field operator to a ringdown wave remains to found, the potential to bridge quantum scattering amplitudes with gravitational wave observations offers one of the most promising paths forward in modern theoretical physics.

5 Outlook

5.1 Future Directions in Gravitational Wave Theory

The future of gravitational wave astronomy relies heavily on the synergy between quantum scattering amplitudes and classical observables. As next-generation observatories like Cosmic Explorer and the Laser Interferometer Space Antenna (LISA) come online, they will detect signals with much higher sensitivity. The demand for high-precision and computationally efficient waveform templates will soon surpass the practical limits of current Numerical Relativity. Successfully extending this formalism to the bound-state ringdown phases represents a critical theoretical frontier. Developing these methods are essential to deepening our understanding of gravity.

5.2 An Undergraduate's Perspective on Theoretical Physics Research

Undertaking theoretical physics research at the undergraduate level is one that pushes you to your limits, but it is entirely possible. Transitioning into this project required learning some Quantum Field Theory, a graduate level course. My hope is that this thesis serves as a stepping stone for future students seeking to enter the rapidly evolving field of classical and quantum gravity.

For students interested in pursuing theory, my recommendation is to do your best to build a solid foundation in the fundamentals. However at the same time, do not be afraid to simply dive in and figure things out as you go. In research, this learn as you go approach is often the only path forward. Even if you have taken every available advanced math and physics course, true research requires navigating what is unknown. To better support students drawn to theory, undergraduate physics curricula must evolve to introduce these advanced theoretical concepts and research opportunities earlier, bridging the gap between standard coursework and modern research. Clearly, it is possible to be apart of cutting edge research without having a PhD in it. From a personal standpoint, completing this project has built my confidence as a researcher. Theoretical physics is a humbling discipline and just when you think you have mastered a concept, a new layer of complexity reveals itself. Yet the true beauty of this work is that the learning never ends.

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